A CORRECTION†

BY

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I have discovered an error (and some minor misstatements) in my paper *A local solution of the difference equation* $\Delta y(x) = F(x)$ *and of related equations* (these Transactions, vol. 39 (1936), pp. 345–379). It is my purpose here to rectify that error. The vulnerable point occurs from relation (7.1) to Theorem 7.1. In order to secure convergence for series (7.2), it was asserted that the Mittag-Leffler theorem could be applied to (7.6), giving a meromorphic function $Z(x; x^*)$. Now the "poles" of this function are at the points

$$x = x^* + \omega_1 + \sum_{j=2}^{k} n_j(\omega_1 - \omega_j),$$

where $n_2, \cdots, n_k$ range independently from $-\infty$ to $+\infty$. But there is no reason to suppose (as I did) that these "poles" have no limit points in the *finite* plane.‡ If there *are* finite limit points, then we cannot conclude that $L[Z]$ is an entire function, and therefore we cannot apply Theorem 6.6 (Carmichael's theorem). Consequently, Theorem 7.1 will not follow from the argument given.

It becomes necessary to rewrite §7. Fortunately it is possible to give a rigorous treatment which is simpler than the old. We now indicate in what way the old paper is to be revised.

(1) In Theorem 4.4, in place of "$\cdots$ *and in this circle* $y(x) \cdots$" read "$\cdots$ *and in the lens-region* $y(x) \cdots$."

(2) In Theorem 4.6, in place of "$\cdots$ *which in this circle satisfies* $\cdots$" read "$\cdots$ *which in a lens-region about* $x=\alpha$ *satisfies* $\cdots$."

(3) §5 should have as heading: "5. **A geometric lemma.**"

(4) Omit all of §5 beginning with the following line (just preceding Lemma 5.2: "We turn now to two lemmas $\cdots$." (This portion is now unnecessary.)

(5) In §6 omit the two paragraphs shortly after (6.17), beginning with "To treat the general case, in which $F(x) \cdots$," and ending with "This implies no loss of generality of equation (6.3)." (This portion is now unnecessary.)

(6) §7 is to be replaced by the following:

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‡ For certain values of $n_2, \cdots, n_k$, the coefficient $b_{n_2,\cdots,n_k}$ is zero, so that the corresponding "pole" does not actually occur. Conceivably, enough "poles" may be absent so that there are no finite limit points, but there seems to be no way of enumerating the "poles" that are present.

7. **The general case.** To treat the general case where $F(x)$ is merely analytic, we modify the method of §4. Instead of directly finding a solution of the homogeneous equation $L[y]=0$, we shall obtain more than one merom rphic solution of equation

$$L[y] = \frac{1}{x-\alpha}. \tag{7.1}$$

Let C be the unique circle, center at x^* and radius ρ^*, assured us by Lemma 5.1. As a consequence of uniqueness there are seen to be precisely two possibilities: Either

Case I. There are at least two points P_j on C, and of these at least one pair, say P_1, P_2, are *diametrically opposite.* Or,

Case II. There are at least three points P_j on C; of these no two are diametrically opposite, but at least one triad of them (say P_1, P_2, P_3) forms an *acute-angled triangle.*

In Case I draw circles of radius σ (any number greater than ρ^*) with centers at P_1, P_2. These circles will form a lens-region enclosing x^*, and as $\sigma\to\rho^*$ the lens-region (including boundary) will close down on x^* as a unique limit point. That is, given any neighborhood $\mathfrak{N}$ of x^*, there exists a σ such that the lens-region of radius σ will lie with its boundary wholly interior to $\mathfrak{N}$.

Now consider Case II. A simple geometric argument will show that of P_1, P_2, P_3, there will be one, say P_1, such that: (i) P_2 and P_3 are on opposite sides of the diameter d_1 through P_1; and (ii) if d_1' is the diameter perpendicular to d_1, then P_2 and P_3 are on that side of d_1' opposite P_1. Now let σ be any number greater than ρ^*, and draw circles of radius σ and centers P_1, P_2, P_3. These circles form a curvilinear triangle and another simple geometric argument will show that as $\sigma\to\rho^*$ this triangle closes down on the single point x^*. Hence again, if $\mathfrak{N}$ is any neighborhood of x^*, there is a σ such that the curvilinear triangle of radius σ lies with its boundary wholly inside $\mathfrak{N}$.

As a solution of (7.1) assume the series

$$Y_1(x;\alpha) = \sum_{n_2,\cdots,n_k=0}^{+\infty} \frac{b_{n_2,\cdots,n_k}}{\left[x-(\alpha+\omega_1)-\sum_{j=2}^{k} n_j(\omega_1-\omega_j)\right]}. \tag{7.2}$$

On substituting into (7.1) we get

$$\frac{1}{x-\alpha} = \sum_{n_2,\cdots,n_k=0}^{+\infty} \frac{\alpha_1 b_{n_2\cdots n_k} + \sum_{j=2}^{k}\alpha_j b_{n_2,\cdots,n_{j-1},n_j-1,n_{j+1},\cdots,n_k}}{\left[x-\alpha-\sum_{j=2}^{k} n_j(\omega_1-\omega_j)\right]},$$

where all b's with a negative subscript are zero. This condition is formally fulfilled if we choose the b's to satisfy

$$\alpha_1 b_{n_2\cdots n_k} + \sum_{j=2}^{k} \alpha_j b_{n_2,\cdots,n_{j-1},n_j-1,n_{j+1},\cdots,n_k} = \begin{cases} 1, & \text{for } n_2 = \cdots = n_k = 0; \\ 0, & \text{for } n_2, \cdots, n_k = 0, 1, 2, \cdots \text{ (but not all zero).} \end{cases} \tag{7.3}$$

We find from $(n_2, \cdots, n_k) = (0, \cdots, 0)$ that $b_{0,\cdots,0} = 1/a_1$. Then on choosing $n_2 + \cdots + n_k = 1$ in all possible ways we find that the $b_{n_2\cdots n_k}$'s $(n_2 + \cdots + n_k = 1)$ are uniquely determined; then the $b_{n_2\cdots n_k}$'s for $n_2 + \cdots + n_k = 2$; etc. That is, there is a *unique set* of b's for which relations (7.3) hold. These values we choose for the coefficients in (7.2).

(7.2) is a *formal* series; there is no reason to suppose that it converges. But if we examine its formal "poles," namely, the points

$$x = \alpha + \omega_1 + \sum_{j=2}^{k} n_j(\omega_1 - \omega_j),$$

we find that they have no limit point in the finite plane.† The classic theorem of Mittag-Leffler is therefore applicable. Set $u = x - \alpha$, so that

$$Y_1(x; \alpha) = \sum_{n_2,\cdots,n_k=0}^{+\infty} \frac{b_{n_2\cdots n_k}}{\left[u - \omega_1 - \sum_{j=2}^{k} n_j(\omega_1 - \omega_j)\right]}.$$

Then there exist polynomials $P_{n_2\cdots n_k}(u)$ such that

$$Z_1(u) = \sum_{n_2,\cdots,n_k=0}^{+\infty} \left(\frac{b_{n_2\cdots n_k}}{\left[u - \omega_1 - \sum_{j=2}^{k} n_j(\omega_1 - \omega_j)\right]} + P_{n_2\cdots n_k}(u) \right) \tag{7.4}$$

defines a meromorphic function whose only poles are at

$$u = \omega_1 + \sum_{j=2}^{k} n_j(\omega_1 - \omega_j)$$

(with corresponding residues $b_{n_2\cdots n_k}$), the series converging uniformly and

† For consider the point P_1 (i.e., $-\omega_1$), which lies on C. If L_1 is the tangent line to C at P_1, then $P_2, \cdots, P_k$ all lie on the same side of L_1. Hence if we start at P_1 and lay off vectors $\sum_{j=2}^{k} n_j(\omega_1 - \omega_j)$, we see (since $n_j \geqq 0$) that the ends of the vectors all lie on this same side of L_1 (save for $n_2 = \cdots = n_k = 0$). If we take components of these vectors in the direction perpendicular to the line L_1, we see that the ends of the vectors go off to infinity as any n_j becomes infinite, so that no finite limit point is possible.

absolutely in every bounded region (the poles in this region being deleted).

On applying the operator L term-wise to the series for Z_1 (as is permissible) we obtain

$$L[Z_1(x-\alpha)] = \sum_{n_2\cdots n_k=0}^{+\infty}\left(\frac{A_{n_2\cdots n_k}}{\left[x-\alpha-\sum_{j=2}^{k} n_j(\omega_1-\omega_j)\right]} + L[P_{n_2\cdots n_k}(x-\alpha)]\right)$$

where $A_{n_2\cdots n_k}$ is the left member of (7.3), so that

$$L[Z_1(x-\alpha)] = \frac{1}{x-\alpha} + \sum_{n_2\cdots n_k=0}^{+\infty} L[P_{n_2\cdots n_k}(x-\alpha)]. \tag{7.5}$$

The right-hand side converges for all x (the point $x=\alpha$ is singular only for the term $1/(x-\alpha)$). That is, the series on the right is an entire function. Now by Theorem 6.6 the equation

$$L[g_1(u)] = \sum_{n_2\cdots n_k=0}^{+\infty} L[P_{n_2\cdots n_k}(u)] \tag{7.6}$$

has an entire function solution $g_1(u)$. Consequently

$$W_1(u) = Z_1(u) - g_1(u) \tag{7.7}$$

is a meromorphic function satisfying the equation

$$L[W_1(x-\alpha)] = \frac{1}{x-\alpha}; \tag{7.8}$$

its only poles are simple poles at the points

$$x = \alpha + \omega_1 + \sum_{j=2}^{k} n_j(\omega_1-\omega_j) \qquad (n_2, \cdots, n_k = 0, 1, 2, \cdots),$$

and the corresponding residues are $b_{n_2\cdots n_k}$ *as given by* (7.3).

In the above work concerning Y_1, Z_1, and W_1, the point P_1 (i.e., the number $-\omega_1$) was preferred over all the others. But in Case I point P_2 is also on C, and in Case II points P_2, P_3 are on C; and these points may equally well be used as was P_1. We thus get, according to the case, one or two unique formal series†

$$Y_2(x;\alpha) = \sum_{n_1,n_3,\cdots,n_k=0}^{+\infty} \frac{c_{n_1n_3\cdots n_k}}{\left[x-(\alpha+\omega_2)-\sum_{j=1,3,\cdots,k} n_j(\omega_2-\omega_j)\right]},$$

† The c's and d's satisfy recurrence relations similar to (7.3).

$$Y_3(x;\alpha) = \sum_{n_1,n_2,n_4,\cdots,n_k=0}^{+\infty} \frac{d_{n_1n_2n_4\cdots n_k}}{\left[x - (\alpha + \omega_3) - \sum_{j=1,2,4,\cdots,k} n_j(\omega_3 - \omega_j)\right]};$$

one or two functions $Z_2(x-\alpha)$, $Z_3(x-\alpha)$; and one or two meromorphic functions $W_2(x-\alpha)$, $W_3(x-\alpha)$ satisfying (respectively)

$$L[W_2(x-\alpha)] = \frac{1}{x-\alpha}, \qquad L[W_3(x-\alpha)] = \frac{1}{x-\alpha},$$

with respective simple poles at

$$x = \alpha + \omega_2 + \sum_{j=1,3,\cdots,k} n_j(\omega_2 - \omega_j), \qquad x = \alpha + \omega_3 + \sum_{j=1,2,4,\cdots,k} n_j(\omega_3 - \omega_j),$$

and residues $c_{n_1n_3\cdots n_k}$, $d_{n_1n_2n_4\cdots n_k}$.

Case I. Here P_1, P_2 are diametrically opposite on C. Let σ be only slightly larger than ρ^*, and with radius σ and centers P_1, P_2, draw a lens-region $\mathfrak{L}$ around x^*. Let the bounding arcs of $\mathfrak{L}$ be C_1, C_2 (C_i being that arc with center at P_i). Let α remain on C_1. Since the point $x^*+\omega_1$ is where the point P_2 would be if C were translated so that x^* falls at the origin, it is seen that as α traverses C_1, $\alpha+\omega_1$ will trace a small arc (in the neighborhood of $x^*+\omega_1$) of a circle of radius σ and center the origin. Hence from our knowledge of the position of the poles

$$x = \alpha + \omega_1 + \sum_{j=2}^{k} n_j(\omega_1 - \omega_j),$$

we can say, if σ is sufficiently close to ρ^*, that $W_1(x-\alpha)$ is analytic about the origin in a circle of radius exceeding ρ^*. That is,

$$W_1(x-\alpha) = \sum_{n=0}^{\infty} B_{1n}(\alpha)x^n, \tag{7.9}$$

where the $B_{1n}(\alpha)$ are analytic functions in the neighborhood of $\alpha=x^*$ (and in particular for α on C_1), and where there is a number $\sigma_1>\rho^*$, *independent of α on C_1*, such that (7.9) converges uniformly for α on C_1 and x in $|x|\leqq\sigma_1$.

An analogous statement applies to $W_2(x-\alpha)$ for α on C_2:

$$W_2(x-\alpha) = \sum_{n=0}^{\infty} B_{2n}(\alpha)x^n, \tag{7.10}$$

uniformly convergent for α on C_2 and x in $|x|\leqq\sigma_2$, where σ_2 is some number exceeding ρ^*. (The $B_{2n}(\alpha)$ are analytic functions in a neighborhood of x^* containing C_2.)

Case II. P_1, P_2, P_3 are on C. Again choosing σ only slightly larger than ρ^*, we obtain a curvilinear triangle $\mathfrak{L}$ of radius σ by drawing arcs C_1, C_2, C_3 with centers P_1, P_2, P_3. For α on C_i the argument used in Case I applies, giving us (7.9), (7.10) or

$$W_3(x-\alpha) = \sum_{n=0}^{\infty} B_{3n}(\alpha)x^n, \tag{7.11}$$

uniformly convergent for α on C_3 and x in $|x| \leqq \sigma_3$, where σ_3 is some number greater than ρ^*. (The $B_{3n}(\alpha)$ are analytic in a neighborhood of x^* containing C_3.)

For any value of i ($i=1$ or 2 in Case I and 1, 2, or 3 in Case II), $\limsup |B_{in}(\alpha)|^{1/n} \leqq 1/\sigma_i \leqq 1/\sigma < 1/\rho^*$, where σ = smallest of σ_1, σ_2, σ_3 and α is on C_i. It follows from Theorem 6.2 and Corollary 6.1 that the series $\sum_{n=0}^{\infty} B_{in}(\alpha)A_n(x)$ converges uniformly for x in some curvilinear polygon $\mathfrak{L}$ about x^* and α on C_i, where $\mathfrak{L}$ can be chosen independent of i. But this series is what we get when we apply L term-wise to the $W_i(x-\alpha)$ series. From this follows

THEOREM 7.1. *According to the case, if a lens-region $\mathfrak{L}$ or a curvilinear triangle $\mathfrak{L}$ be drawn*† *about x^*, with radius σ sufficiently near to ρ^*, then*

$$\frac{1}{x-\alpha} = \sum_{n=0}^{\infty} B_{jn}(\alpha)A_n(x), \qquad \begin{cases} j = 1, 2 & \text{in Case I}, \\ j = 1, 2, 3 & \text{in Case II}, \end{cases} \tag{7.11}$$

the convergence being uniform in x and α for α on C_j and x in some neighborhood $\mathfrak{N}$ of x^*. ($\mathfrak{N}$ can be chosen independent of j.)

Now let $F(x)$ be analytic about $x=x^*$. Then there exists a lens or triangle $\mathfrak{L}$ around x^* lying (together with its boundary) wholly in the region of analyticity of $F(x)$, and with radius so close to ρ^* that Theorem 7.1 applies. Let x be in the region $\mathfrak{N}$ of Theorem 7.1. Multiply (7.11) by $F(\alpha)$ and integrate over $\mathfrak{L}$. This gives

$$F(x) = \sum_{n=0}^{\infty} f_n A_n(x), \tag{7.12}$$

where

$$f_n = -\sum_{\substack{j=1,2 \\ \text{or } j=1,2,3}} \frac{1}{2\pi i}\int_{C_j} F(\alpha)B_{jn}(\alpha)d\alpha; \tag{7.13}$$

and (7.12) converges uniformly in $\mathfrak{N}$. We thus have

† The points P_j will of course be the centers of the arcs forming $\mathfrak{L}$.

THEOREM 7.2. *If $F(x)$ is analytic about $x=x^*$, it has a convergent A_n-expansion, given by* (7.12).

Combining this with Theorem 6.1:

THEOREM 7.3. *A necessary and sufficient condition that a function $F(x)$ have an (convergent) A_n-expansion is that it be analytic at $x=x^*$.*

By Theorem 6.2, $\limsup |f_n|^{1/n} < 1/\rho^*$, so that the series

$$y(x) = \sum_0^\infty f_n x^n \tag{7.14}$$

converges in $|x| < \rho^* + \epsilon$, for some $\epsilon > 0$. On applying L to (7.17) we get

$$L[y(x)] = \sum f_n L[x^n] = \sum f_n A_n(x) = F(x),$$

so that we have

THEOREM 7.4. *If $F(x)$ is analytic about $x=x^*$, then the function $y(x)$ of* (7.14) *is analytic in a circle (about $x=0$) of radius greater than ρ^*, and for all x in a sufficiently small curvilinear polygon (about $x=x^*$) $y(x)$ satisfies the equation*

$$L[y(x)] = F(x). \tag{6.3}$$

The point x^* is of course significant for A_n-expansions, but not for equation (6.3). For let $F(x)$ be analytic about $x=c$, and define $G(x)=F(x+c-x^*)$. $G(x)$ is analytic about $x=x^*$ and therefore there exists a function $z(x)$, analytic at x^*, such that $L[z(x)]=G(x)$. Consequently, the function $y(x) = z(x-c+x^*)$ satisfies $L[y(x)]=F(x)$, and we have the final

THEOREM 7.5. *If $F(x)$ is analytic about $x=c$, there exists a function $y(x)$, analytic about $x=c-x^*$ in a circle of radius exceeding ρ^*, such that for all x in a sufficiently small neighborhood (curvilinear polygon) of $x=c$, $y(x)$ satisfies equation* (6.3).

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